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On a new non-isospectral variant of the Boussinesq hierarchy

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Abstract. We give a new non-isospectral extension to 2 + 1 dimensions of the Boussinesq hierarchy. Such a non-isospectral extension of the third-order scattering problem $\psi_{xxx}+U\psi_x+(V-\lambda)\psi=0$ has not been considered previously. This extends our previous results on one-component hierarchies in 2 + 1 dimensions associated to third-order non-isospectral scattering problems. We characterize our entire (2 + 1)-dimensional hierarchy and its linear problem using a single partial differential equation and its corresponding non-isospectral scattering problem. This then allows an alternative approach to the construction of linear problems for the entire (2 + 1)-dimensional hierarchy. Reductions of this hierarchy yield new integrable hierarchies of systems of ordinary differential equations together with their underlying linear problems. In particular, we obtain a 'fourth Painlevé hierarchy', i.e. a hierarchy of ordinary differential equations having the fourth Painlevé equation as its first member. We also obtain a hierarchy having as its first member a generalization of an equation defining a new transcendent due to Cosgrove.

1. Introduction

Higher-dimensional extensions of completely integrable equations in 1 + 1 dimensions can be obtained in various ways. Here we are interested in one particular kind of extension, namely that to equations having non-isospectral scattering problems. The first such example appears to be due to Calogero [1], and has as a special case the partial differential equation (PDE)

$$U_t = \mathcal{R}U_y \tag{1}$$

where $\mathcal{R} = \partial_x^2 + 4U + 2U_x\partial_x^{-1}$ is the recursion operator of the Korteweg–de Vries (KdV) hierarchy [1, 2] (here $\partial_x \equiv \partial/\partial x$ and similarly in what follows for ∂_y). The application of the inverse scattering transform to this equation has been discussed in [1, 3]. More recently, it has been shown to admit 'breaking soliton' solutions [4].

Similar (2 + 1)-dimensional non-isospectral extensions of other well known PDEs have also been given, for example, for the nonlinear Schrödinger equation [5–7], the classical Boussinesq system [8] and the Fuchssteiner–Fokas–Camassa–Holm equation [9]. However, all of these examples are based on second-order scattering problems. Recently, the present authors have given non-isospectral extensions in 2 + 1 dimensions of one-component hierarchies of PDEs associated to third-order (Sawada–Kotera and Kaup–Kupershmidt) scattering problems [10]. Here we extend this work still further by constructing a non-isospectral variant of the Boussinesq hierarchy in 2 + 1 dimensions. Again these results are new. Reductions of the resulting two-component hierarchy of PDEs to ordinary differential equations (ODEs) then give new hierarchies of integrable systems of ODEs together with their underlying linear problems. These include a hierarchy of coupled ODEs which has as its first member a system equivalent to the fourth Painlevé equation P_{IV}; that is, we obtain a P_{IV} hierarchy. We also

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obtain a hierarchy of coupled ODEs which at lowest order gives a generalization of an ODE due to Cosgrove [11].

The layout of the paper is as follows. In section 2 we construct our $(2+1)$-dimensional extension of the Boussinesq hierarchy together with the corresponding hierarchy of linear problems. This construction is based on a characterization of the entire hierarchy and its scattering problem using a single equation and its non-isospectral scattering problem. In section 3 we discuss reductions of this hierarchy to one component, and show how this allows us to recover our previous results. In section 4 we discuss reductions to systems of ODEs. Section 5 is devoted to consideration of an explicit example. In section 6 we give a summary and conclusions.

2. A non-isospectral Boussinesq hierarchy

Motivated by (1) and the work in [10], we begin by considering the $(2+1)$-dimensional system

$$\boldsymbol{U}_t = \mathcal{R}\boldsymbol{U}_\tau + \boldsymbol{G} \tag{2}$$

where $\boldsymbol{U} = (U, V)^T$, $\mathcal{R}$ is the recursion operator of the Boussinesq hierarchy, $\boldsymbol{G} = (0, g)^T$, and g is an arbitrary function of t and τ which is introduced by our non-isospectral condition (the equation satisfied by the spectral parameter in the corresponding Lax pair). For an appropriate choice of coordinates, the recursion operator of the Boussinesq hierarchy can be written as [12–16]

$$\mathcal{R} = J_0 J_1^{-1} \tag{3}$$

where the Hamiltonian operators J_0 and J_1 are defined by

$$J_0 = \begin{pmatrix} 2\partial_x^3 + 2U\partial_x + U_x & -\partial_x^4 - \partial_x^2 U + 3\partial_x V - V_x \\ \partial_x^4 + U\partial_x^2 + 3V\partial_x + V_x & -\frac{1}{3}\big[2\partial_x^5 + 2(U\partial_x^3 + \partial_x^3 U) + (U^2 - 3V_x)\partial_x \\ & \quad +\partial_x(U^2 - 3V_x)\big] \end{pmatrix} \tag{4}$$

and

$$J_1 = 3\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{5}$$

(here we follow the choice of coordinates used in [16]).

The system (2) has the Lax pair

$$\psi_{xxx} = -U\psi_x - (V - \lambda)\psi \tag{6}$$

$$\psi_t = \lambda\psi_\tau + \tfrac{1}{3}\big(\partial_x^{-1}U_\tau\big)\psi_{xx} + \tfrac{1}{3}\big(\partial_x^{-1}V_\tau - U_\tau\big)\psi_x + \tfrac{1}{9}\big(2U\partial_x^{-1}U_\tau - 3V_\tau + 2U_{x\tau}\big)\psi \tag{7}$$

where the spectral parameter $\lambda = \lambda(\tau, t)$ satisfies

$$\lambda_t = \lambda\lambda_\tau + g. \tag{8}$$

The system (2) and the corresponding non-isospectral scattering problem (6)–(8) are new.

We now use the system (2) and its associated non-isospectral scattering problem to generate our $(2+1)$-dimensional Boussinesq hierarchy and at the same time the corresponding hierarchy of non-isospectral scattering problems. We do this by observing that, for suitably specified flow times t and τ, equation (2) can be understood as representing a generic member of this hierarchy and (6)–(8) its underlying scattering problem. We then use these equations to iterate between both successive flows and their linear problems. We also iterate on the function $\boldsymbol{G}$.

This provides an alternative approach to that of seeking expansions in λ for the coefficients of the temporal part of the Lax pair (a technique originally proposed in [17]). As the starting point for this iteration (i.e. as the base equation) we take the system

$$\boldsymbol{U}_{t_1} = \mathcal{R}\boldsymbol{U}_y + aJ_0\begin{pmatrix}0\\1\end{pmatrix} + \mathcal{R}\boldsymbol{G}_0 + \boldsymbol{G}_1. \tag{9}$$

When performing our iteration each $\boldsymbol{G}_i = (0, g_i)^T$, a and λ are considered to be functions of all possible flow times t_j and y, but not of x.

The reason for making this choice of base equation is in order that reductions to 1 + 1 dimensions will include both sequences (see [13, 15, 16]) of the Boussinesq hierarchy. These two sequences, with flow times τ_m, $m = 0, 1, 2, \ldots$, are defined as

$$\boldsymbol{U}_{\tau_m} = J_0 M_m[\boldsymbol{U}] \qquad M_{m+2}[\boldsymbol{U}] = J_1^{-1}J_0 M_m[\boldsymbol{U}] \tag{10}$$

where $M_0[\boldsymbol{U}] = (1, 0)^T$ and $M_1[\boldsymbol{U}] = (0, 1)^T$. We see that the second term on the right-hand side of (9) is a copy of the τ_1 flow, and it is this that will allow us to include reductions to the second (τ_{odd}) sequence of the Boussinesq hierarchy. This τ_1 flow can be written as a scalar equation, and is of course the Boussinesq equation itself,

$$U_{\tau_1\tau_1} = -\tfrac{1}{3}(U_{xx} + 2U^2)_{xx}. \tag{11}$$

The system (9) can be written locally by setting $U = u_x$, $V = v_x$,

$$\begin{aligned} u_{xt_1} = {} & \tfrac{1}{3}[2v_{xxxy} - u_{xxxxy} - u_x u_{xxy} - u_y u_{xxx} + 2u_x v_{xy} - 2u_{xx}u_{xy} + 3v_x u_{xy} \\ & + 2u_y v_{xx} + v_y u_{xx}] + a[2v_{xx} - u_{xxx}] + \tfrac{1}{3}g_0[2u_x + xu_{xx}] \end{aligned} \tag{12}$$

$$\begin{aligned} v_{xt_1} = {} & \tfrac{1}{9}[3v_{xxxxy} - 2u_{xxxxxy} - 2u_y u_{xxxx} - 4u_x u_{xxxy} + 3u_x v_{xxy} - 6u_{xx}u_{xxy} \\ & - 6u_{xy}u_{xxx} + 3u_y v_{xxx} - 2u_x^2 u_{xy} + 9v_x v_{xy} + 3v_y v_{xx} + 6v_{xx}u_{xy} - 2u_x u_y u_{xx}] \\ & - \tfrac{1}{3}a[2u_{xxxx} + 2u_x u_{xx} - 3v_{xxx}] + \tfrac{1}{3}g_0[3v_x + xv_{xx}] + g_1. \end{aligned} \tag{13}$$

This system has the Lax pair

$$\psi_{xxx} = -u_x\psi_x - (v_x - \lambda)\psi \tag{14}$$

$$\begin{aligned} \psi_{t_1} = {} & \lambda\psi_y + \tfrac{1}{3}(u_y + 3a)\psi_{xx} + \tfrac{1}{3}(v_y - u_{xy} + g_0 x)\psi_x \\ & + \tfrac{1}{9}[(2u_x u_y - 3v_{xy} + 2u_{xxy}) + 6au_x - 3g_0]\psi \end{aligned} \tag{15}$$

where the spectral parameter λ satisfies the constraint

$$\lambda_{t_1} = \lambda\lambda_y + \lambda g_0 + g_1. \tag{16}$$

In order to use (2) to iterate between successive flows of our hierarchy, and their corresponding scattering problems, we begin by writing a generic member of this (2 + 1)-dimensional hierarchy as

$$\boldsymbol{U}_{t_n} = \boldsymbol{K}_n \tag{17}$$

and the corresponding generic evolution equations for the eigenfunction ψ and the spectral parameter λ as

$$\psi_{t_n} = \Gamma_n\psi_y + Q_n\psi_{xx} + (P_n - Q_{n,x})\psi_x + \tfrac{1}{3}(2UQ_n - 3P_{n,x} + 2Q_{n,xx})\psi \tag{18}$$

$$\lambda_{t_n} = \Lambda_n. \tag{19}$$

In equation (18) we follow the notation used in [16], although the form of (18) can clearly be motivated by that of (7). We then obtain from (2), (7) and (8) the recursion relations

$$\boldsymbol{K}_n = \mathcal{R}\boldsymbol{K}_{n-1} + \boldsymbol{G}_n \tag{20}$$

$$\Gamma_n = \lambda\Gamma_{n-1} \tag{21}$$

$$\Lambda_n = \lambda\Lambda_{n-1} + g_n \tag{22}$$

$$\boldsymbol{P}_n = \lambda\boldsymbol{P}_{n-1} + J_1^{-1}\boldsymbol{K}_{n-1} \tag{23}$$

where $\boldsymbol{P}_n = (P_n, Q_n)^T$ and $\boldsymbol{G}_n = (0, g_n)^T$. These recursion relations, together with the base equation (9) and its scattering problem and corresponding constraint on λ, then yield the hierarchy of evolution equations

$$\boldsymbol{U}_{t_n} = \boldsymbol{K}_n = \mathcal{R}^n\boldsymbol{U}_y + a\mathcal{R}^{n-1}J_0M_1[\boldsymbol{U}] + \sum_{i=0}^{n}\mathcal{R}^{n-i}\boldsymbol{G}_i \tag{24}$$

and corresponding hierarchy of spectral problems

$$\psi_{xxx} = -U\psi_x - (V - \lambda)\psi \tag{25}$$

$$\psi_t = \lambda^n\psi_y + Q_n\psi_{xx} + (P_n - Q_{n,x})\psi_x + \tfrac{1}{3}(2UQ_n - 3P_{n,x} + 2Q_{n,xx})\psi. \tag{26}$$

Here $\boldsymbol{P}_n = (P_n, Q_n)^T$ is given by

$$\boldsymbol{P}_n = \begin{pmatrix} 0 \\ a \end{pmatrix}\lambda^{n-1} + J_1^{-1}\sum_{i=0}^{n-1}\lambda^{n-i-1}\boldsymbol{K}_i \tag{27}$$

where we have set $\boldsymbol{K}_0 = \boldsymbol{U}_y + \boldsymbol{G}_0$, and λ satisfies

$$\lambda_{t_n} = \lambda^n\lambda_y + \sum_{i=0}^{n}\lambda^{n-i}g_i. \tag{28}$$

This hierarchy of evolution equations (24) in 2 + 1 dimensions is new. The first term on the right-hand side of (24) represents a non-isospectral extension to 2 + 1 dimensions of the first sequence (τ_{even}) of the Boussinesq hierarchy (10). Such an extension has not been considered before. The second term is the $(1 + 1)$-dimensional sequence which includes the Boussinesq equation itself. The third term represents a non-isospectral deformation which gives rise to non-autonomous terms, which for $n > 2$ are in the general case non-local. We note that allowing τ in (2) to be a vector would allow us to obtain linear combinations of the flows (24).

Reductions of the system (24) to PDEs in 1 + 1 dimensions include the non-isospectral deformations of standard Boussinesq flows considered in [15] ($\partial_y = \partial_x$), and also reductions to non-isospectral deformations of inverse Boussinesq flows ($\partial_{t_n} = 0$). We note that a discussion of non-isospectral scattering in 1 + 1 dimensions can also be found in [18].

3. Reductions in components

The hierarchy (24) is a two-component hierarchy of PDEs in 2 + 1 dimensions. We now consider reductions of this hierarchy to scalar equations. We find that the even flows $n = 2m$ of this hierarchy admit both of the standard reductions $(U, V) = (2W, W_x)$ and $(U, V) = (W/2, 0)$ of the third-order scattering operator (6). These reductions then yield the hierarchies of one-component equations in 2 + 1 dimensions—non-isospectral extensions of the Kaup–Kupershmidt and Sawada–Kotera hierarchies, respectively—obtained in [10]. The odd flows $n = 2m - 1$ of (24) also admits the reduction $(U, V) = (2W, W_x)$, which then gives a hierarchy of scalar PDEs in 1 + 1 dimensions. This scalar hierarchy can also be found in [10].

In order to show the above we consider the form of the square of the recursion operator $\mathcal{R}$ in each of these two reductions. In the case $(U, V) = (2W, W_x)$ we find

$$\mathcal{R}^2 = \begin{pmatrix} -\frac{1}{27}\widehat{\mathcal{R}} & 0 \\ \frac{1}{54}\partial_x(K[W]\theta[W] - \theta[W]K[W]) & -\frac{1}{27}\partial_x K[W]\theta[W]\partial_x^{-1} \end{pmatrix} \tag{29}$$

where

$$\theta[W] = \partial_x^3 + W\partial_x + \partial_x W \tag{30}$$

$$K[W] = \partial_x^{-1}\left[\partial_x^5 + 3\left(\partial_x W\partial_x^2 + \partial_x^2 W\partial_x\right) + 2\left(\partial_x^3 W + W\partial_x^3\right) + 8\left(\partial_x W^2 + W^2\partial_x\right)\right]\partial_x^{-1} \tag{31}$$

and $\widehat{\mathcal{R}} = \theta[W]K[W]$ is the recursion operator of the Kaup–Kupershmidt hierarchy [19, 20]. In the case $(U, V) = (W/2, 0)$ we find that the square of the recursion operator of the Boussinesq hierarchy is of the form

$$\mathcal{R}^2 = \begin{pmatrix} -\frac{1}{27}\widetilde{\mathcal{R}} & \bullet \\ 0 & \bullet \end{pmatrix} \tag{32}$$

where $\widetilde{\mathcal{R}}$ is the recursion operator of the Sawada–Kotera hierarchy [19, 20].

It follows that if for each of the above reductions we consider the even flows $n = 2m$ of the hierarchy (24) with all $g_{2k} = 0$, $k = 0, 1, 2, \ldots, m$, and also rescale $\partial_{t_{2m}} \to \left(-\frac{1}{27}\right)^m \partial_{t_{2m}}$, we obtain

$$W_{t_{2m}} = \widehat{\mathcal{R}}^m W_y + 3a\widehat{\mathcal{R}}^{m-1}\theta[W](W_{xx} + 4W^2) - \sum_{i=1}^{m} q_i\widehat{\mathcal{R}}^{m-i}\theta[W]\,x \tag{33}$$

and

$$W_{t_{2m}} = \widetilde{\mathcal{R}}^m W_y + 3a\widetilde{\mathcal{R}}^{m-1}\theta[W]\left(W_{xx} + \tfrac{1}{4}W^2\right) - \sum_{i=1}^{m} q_i\widetilde{\mathcal{R}}^{m-i}\theta[W]\,x \tag{34}$$

respectively, where $q_i = -\frac{1}{3}(-27)^i g_{2i-1}$. These are the one-component (2 + 1)-dimensional hierarchies presented in [10]. For the special case of the second sequence of the standard autonomous (1 + 1)-dimensional Boussinesq hierarchy ($\partial_y = 0$ and all $g_i = 0$ in (24)) these results can be found in [14].

We now consider the odd flows $n = 2m - 1$ of (24) in the case where $\partial_{t_n} = 0$ and $g_{2k} = 0$, $k = 0, 1, 2, \ldots, m - 1$, and make the reduction $(U, V) = (2W, W_x)$, again setting $q_i = -\frac{1}{3}(-27)^i g_{2i-1}$. Since for $n = 1$ in (24) this obtains

$$0 = \mathcal{R}\boldsymbol{U}_y + aJ_0M_1[\boldsymbol{U}] + \boldsymbol{G}_1 = \left(0, \partial_x K[W]W_y + 3a\partial_x(W_{xx} + 4W^2) - q_1\right)^T \tag{35}$$

we then find, using (29), that the hierarchy (24) reduces to

$$\partial_x(K[W]\theta[W])^{m-1}K[W]W_y + 3a\partial_x(K[W]\theta[W])^{m-1}(W_{xx} + 4W^2)$$
$$-\partial_x\sum_{i=1}^{m} q_i\left(K[W]\theta[W]\right)^{m-i}x = 0. \tag{36}$$

This one-component hierarchy in 1 + 1 dimensions can also be found in [10].

4. Reductions to systems of ODEs

We now consider reductions of the hierarchy (24) to systems of ODEs. We take $\partial_{t_n} = 0$ and $\partial_y = \alpha\partial_x$, for some constant α, which then yields

$$\alpha\mathcal{R}^n \boldsymbol{U}_x + a\mathcal{R}^{n-1} J_0 M_1[\boldsymbol{U}] + \sum_{i=0}^{n} \mathcal{R}^{n-i}\boldsymbol{G}_i = 0 \tag{37}$$

where a and all g_i are now constant parameters. Setting $a = 0$ obtains non-autonomous extensions of the stationary flows of the first sequence (τ_{even}) of the Boussinesq hierarchy; setting $\alpha = 0$ gives non-autonomous extensions of the stationary flows of the second sequence (τ_{odd}) of the Boussinesq hierarchy. Here for reasons of convenience we consider both of these together. Further generalizations of (37) are readily obtained by adding lower-order Boussinesq flows. Following the approach in [21] we are able to use our non-isospectral scattering problems to obtain linear problems for the hierarchy of ODEs (37). Thus we obtain

$$\psi_{xxx} = -U\psi_x - (V-\lambda)\psi \tag{38}$$

$$\left(\sum_{i=0}^{n} \lambda^{n-i} g_i\right)\psi_\lambda = Q_n\psi_{xx} + (\alpha\lambda^n + P_n - Q_{n,x})\psi_x + \tfrac{1}{3}(2UQ_n - 3P_{n,x} + 2Q_{n,xx})\psi \tag{39}$$

where we assume that not all g_i are zero. Here $\boldsymbol{P}_n = (P_n, Q_n)^T$ is given by (27) where now

$$\boldsymbol{K}_i = \alpha\mathcal{R}^i \boldsymbol{U}_x + a\mathcal{R}^{i-1} J_0 M_1[\boldsymbol{U}] + \sum_{j=0}^{i} \mathcal{R}^{i-j}\boldsymbol{G}_j \tag{40}$$

and $\boldsymbol{K}_0 = \alpha\boldsymbol{U}_x + \boldsymbol{G}_0$.

In the local case $g_i = 0$, $i = 0, 1, 2, \ldots, n-2$, the hierarchy (37) reads

$$\alpha\mathcal{R}^n \boldsymbol{U}_x + a\mathcal{R}^{n-1} J_0 M_1[\boldsymbol{U}] + \tfrac{1}{3}g_{n-1}\begin{pmatrix} 2U + xU_x \\ 3V + xV_x \end{pmatrix} + g_n\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{41}$$

where we now assume that at least one of g_n, g_{n-1} is non-zero†. We note that in the current work we do not address the question of the order of the irreducible system equivalent to (41); for example, in the case $n = 1$ with $\alpha = 0$, this system is fourth order but is in fact equivalent to the fourth Painlevé equation P_{IV}, which is of course second order (see section 5 and [22] for details). Any integrations of the system (41) can be used (if we so wish) to eliminate higher-order derivatives of U and V in the above linear problem. Similarly for the corresponding matrix linear problem, which given the above scalar linear problem we can easily write down.

When $g_n = 0$ the hierarchy (41) is a similarity reduction of sums of Boussinesq flows: when $a = 0$ it is the reduction of the first Boussinesq sequence under

$$\begin{aligned} U &= \frac{P(X)}{[(3n+1)\gamma t_n]^{2/(3n+1)}} \\ V &= \frac{Q(X)}{[(3n+1)\gamma t_n]^{3/(3n+1)}} \\ X &= \frac{x}{[(3n+1)\gamma t_n]^{1/(3n+1)}} \end{aligned} \tag{42}$$

† Note that if we had taken $U_{t_3} = \mathcal{R}U_y + \mathcal{R}g_0 + g_1$ as the base equation for the non-isospectral KdV hierarchy discussed in [10], then this hierarchy would have read $U_{t_{2n+1}} = \mathcal{R}^n U_y + \sum_{i=0}^{n}\mathcal{R}^{n-i}g_i$, and in the local case $g_i = 0$, $i = 0, 1, \ldots, n-2$, of our reductions to ODEs we would have obtained $\mathcal{R}^n U_x + g_{n-1}(4U + 2xU_x) + g_n = 0$ for all $n > 0$, i.e. the same structure as (41), with one sequence.

and for $\alpha = 0$ it is the reduction of the second Boussinesq sequence under

$$U = \frac{P(X)}{[(3n-1)\gamma t_n]^{2/(3n-1)}} \qquad V = \frac{Q(X)}{[(3n-1)\gamma t_n]^{3/(3n-1)}} \qquad X = \frac{x}{[(3n-1)\gamma t_n]^{1/(3n-1)}} \tag{43}$$

where in each case $\gamma = g_{n-1}/3$.

When $g_{n-1} = 0$ each component of the hierarchy (41) is easily integrated to obtain a system with two constants of integration. We can then give a matrix Lax pair with compatibility condition this integrated system. Our assumption in this case that $g_n \neq 0$ allows us to remove one constant of integration, if we so wish.

Thus far we have been insisting that not all g_i vanish. If, however, we take all $g_i = 0$ we can, following [23], use the linear system (38) and (39)—or equivalently the above-mentioned matrix linear problem—to obtain first integrals of this integrated version of (41) (with $g_n = g_{n-1} = 0$). This is done in the next section for the case $n = 1$.

The two-component hierarchy of ODEs (37) admits the same reductions to scalar hierarchies as obtained in the previous section for our (2+1)-dimensional Boussinesq hierarchy. That is, the even flows $n = 2m$ of (37) admit both of the reductions $(U, V) = (2W, W_x)$ and $(U, V) = (W/2, 0)$ to the corresponding ODE reductions of (33) and (34), which are, respectively,

$$\alpha\widehat{\mathcal{R}}^m W_x + 3a\widehat{\mathcal{R}}^{m-1}\theta[W](W_{xx} + 4W^2) - \sum_{i=1}^{m} q_i\widehat{\mathcal{R}}^{m-i}\theta[W]\,x = 0 \tag{44}$$

and

$$\alpha\widetilde{\mathcal{R}}^m W_x + 3a\widetilde{\mathcal{R}}^{m-1}\theta[W]\big(W_{xx} + \tfrac{1}{4}W^2\big) - \sum_{i=1}^{m} q_i\widetilde{\mathcal{R}}^{m-i}\theta[W]\,x = 0. \tag{45}$$

The odd flows $n = 2m-1$ of (37) allow the reduction $(U, V) = (2W, W_x)$ to the corresponding ODE reduction of (36), i.e.

$$\alpha\partial_x H_{2m}[W] + 3a\partial_x H_{2m-1}[W] - \partial_x \sum_{i=1}^{m} q_i (K[W]\theta[W])^{m-i} x = 0 \tag{46}$$

where $H_m[W]$ are defined by the recursion relations $H_{m+2}[W] = K[W]\theta[W]H_m[W]$, and $H_0[W] = 1$, $H_1[W] = W_{xx} + 4W^2$ [19, 20]. These one-component hierarchies of ODEs (44), (45) and (46) can be found in [10].

5. Example: the case $n = 1$

Here we consider in more detail the case $n = 1$ of the above hierarchies of PDEs and ODEs. The case $n = 1$ of our (2 + 1)-dimensional Boussinesq hierarchy (24) is given by equations (12) and (13) (with $U = u_x$, $V = v_x$). This admits the reduction $(U, V) = (2W, W_x)$ $((u, v) = (2w, w_x))$ to the scalar equation $\partial_x K[W]W_y + 3a\partial_x H_1[W] - q_1$. This is the case $m = 1$ of the hierarchy (36), and is written locally as

$$0 = w_{xxxxxy} + 10w_{xxxy}w_x + 2w_{xxxx}w_y + 15w_{xxy}w_{xx} + 9w_{xxx}w_{xy} + 16w_{xy}w_x^2 + 16w_{xx}w_xw_y + 3a\big(w_{xxxx} + 8w_xw_{xx}\big) - q_1. \tag{47}$$

This subequation of an inverse Boussinesq flow can also be obtained as a subequation of an inverse Kaup–Kupershmidt flow [10], but does not seem to have been considered in the literature prior to the work of the current authors. We note that the function a can always be removed from (12), (13) and (47) by a simple shift. A Bäcklund transformation for (47) is given in [24].

The case $n = 1$ of the hierarchy (37), obtained as a reduction of (24), is $\alpha\mathcal{R}\boldsymbol{U}_x + aJ_0M_1[\boldsymbol{U}] + \mathcal{R}\boldsymbol{G}_0 + \boldsymbol{G}_1 = 0$, i.e.

$$\tfrac{1}{3}\alpha\big(2V_{xx} - U_{xxx} + 4VU - 2UU_x\big)_x + a\big(2V - U_x\big)_x + \tfrac{1}{3}g_0\big(2U + xU_x\big) = 0 \tag{48}$$

$$\begin{aligned}&\tfrac{1}{3}\alpha\big(V_{xxx} - \tfrac{2}{3}U_{xxxx} - 2UU_{xx} + 2UV_x - U_x^2 + 2V^2 - \tfrac{4}{9}U^3\big)_x \\ &\qquad -\tfrac{1}{3}a\big(2U_{xx} - 3V_x + U^2\big)_x + \tfrac{1}{3}g_0\big(3V + xV_x\big) + g_1 = 0.\end{aligned} \tag{49}$$

For this system we obtain a linear problem with ψ_λ given by

$$\begin{aligned}(\lambda g_0 + g_1)\psi_\lambda = {}& \tfrac{1}{3}\big(\alpha U + 3a\big)\psi_{xx} + \tfrac{1}{3}\big[\alpha\big(V - U_x + 3\lambda\big) + g_0 x\big]\psi_x \\ &+ \tfrac{1}{9}\big[\alpha\big(2U_{xx} + 2U^2 - 3V_x\big) + 6aU - 3g_0\big]\psi\end{aligned} \tag{50}$$

where we assume that at least one of g_0, g_1 is non-zero. Taking $a = 0$ then gives the first member of a hierarchy of systems of ODEs obtained as non-autonomous extensions of the stationary flows of the first Boussinesq sequence. Taking $\alpha = 0$ gives the first member of a hierarchy of systems of ODEs obtained as non-autonomous extensions of the stationary flows of the second Boussinesq sequence.

For $\alpha = 0$ the above system is therefore a non-autonomous extension of the stationary flow of the Boussinesq system itself. In the case $g_0 \neq 0$ elimination of V gives (assuming $a \neq 0$) a fourth-order ODE for U,

$$a^2\big(U_{xx} + 2U^2\big)_{xx} + \tfrac{1}{3}g_0^3\big(x^2U_{xx} + 7xU_x + 8U\big) = 0 \tag{51}$$

which can be obtained from the Boussinesq system under the similarity reduction (43) and is equivalent to the fourth Painlevé equation P_{IV} [22]. (Note that for this case of the system (48) and (49) we can always set $g_1 = 0$ by a simple shift on V.) We note that P_{IV} does of course have a well known second-order linear problem [25–27]. However, here it appears at the base of a hierarchy of systems of ODEs for which we are able to give third-order linear problems. This hierarchy (i.e. (41) with $\alpha = 0$, to which we can readily add lower-order Boussinesq flows) can then be referred to as a P_{IV} hierarchy. In the case $\alpha = 0$ of (48) and (49) having $g_0 = 0$, elimination of V gives the first Painlevé equation P_I (since we then assume $g_1 \neq 0$). Once again this equation has a well known second-order linear problem [25–27].

For $\alpha \neq 0$ the system (48) and (49) is the first member of a hierarchy of systems of ODEs having third-order linear problems related to the first Boussinesq sequence. This hierarchy is (41) with $a = 0$. However, since we can always add lower-order Boussinesq flows to this hierarchy, in what follows we include the parameter a in our consideration of (48) and (49). As we shall see later, this system includes as a special case an ODE defining a new transcendent due to Cosgrove [11]. Thus our hierarchy is, in fact, based on a generalization of Cosgrove's equation.

Cosgrove's equation can be obtained from the special case $g_0 = 0$ and so, for reasons of simplicity, we take this restriction in the discussion of (48) and (49) we give here. In this case the system (48) and (49) is equivalent to

$$0 = \alpha\big(2V_{xx} - U_{xxx} + 4VU - 2UU_x\big) + 3a\big(2V - U_x\big) - 3C \tag{52}$$

$$0 = \alpha\big(V_{xxx} + 2UU_{xx} + 2UV_x - U_x^2 + 8U_xV - 6V^2 + \tfrac{4}{3}U^3\big) + 3a\big(V_x + U^2\big) - 9g_1x + 9D \tag{53}$$

where C and D are two constants of integration, and where we assume $g_1 \neq 0$. This system has the matrix linear problem

$$\boldsymbol{\Psi}_x = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}U & 0 & 1 \\ V - \frac{1}{2}U_x - \lambda & -\frac{1}{2}U & 0 \end{pmatrix} \boldsymbol{\Psi} \tag{54}$$

$$9g_1\boldsymbol{\Psi}_\lambda = \begin{pmatrix} \begin{array}{c} \alpha\big(U_{xx} - 3V_x - \frac{1}{2}U^2\big) \\ -\frac{3}{2}aU \end{array} & 3\alpha\big(V + 3\lambda\big) & -3(\alpha U + 3a) \\ \begin{array}{c} -\alpha\big[V_{xx} + \frac{3}{2}UU_x \\ +\frac{1}{2}UV + \frac{3}{2}\lambda U\big] \\ +3a\big(3\lambda - V\big) - 3C \end{array} & \alpha\big(U_{xx} + U^2\big) + 3aU & 3\alpha\big[3\lambda + V - U_x\big] \\ \begin{array}{c} \alpha\big[\frac{1}{2}UU_{xx} - U_x^2 \\ +3VU_x - 3V^2 + \frac{7}{12}U^3 \\ +3\lambda(2V - U_x) - 9\lambda^2\big] \\ +\frac{3}{4}aU^2 - 9g_1x + 9D \end{array} & \begin{array}{c} \alpha\big[V_{xx} + \frac{7}{2}UV - \frac{3}{2}\lambda U\big] \\ -6C + 3a\big(3\lambda + V\big) \end{array} & \begin{array}{c} \alpha\big[3V_x - 2U_{xx} \\ -\frac{1}{2}U^2\big] - \frac{3}{2}aU \end{array} \end{pmatrix} \boldsymbol{\Psi}. \tag{55}$$

Note that taking $\alpha = 0$ yields the above-mentioned reduction of (48) and (49) to P_I.

Our assumption thus far that $g_1 \neq 0$ allows us to set $D = 0$ in the system (52) and (53), and in the above matrix linear problem, if we so wish. If we now, however, take $g_1 = 0$, we can then use this matrix linear problem to obtain first integrals of the resulting autonomous version of (52) and (53). Taking the determinant of the matrix in (55) (with $g_1 = 0$) yields a quartic in λ of the form

$$\mathrm{Det} = -\alpha^3\lambda^4 + (\alpha^2 D - a^3)\lambda^2 + A\lambda + B. \tag{56}$$

Here A and B are two constants of motion of the system (52) and (53) with $g_1 = 0$. We find that

$$\begin{aligned} A = \big(\tfrac{1}{3}\alpha\big)^3\big[&2V_xV_{xx} - U_{xx}V_{xx} - 2UU_xU_{xx} + 2UU_xV_x + 4UVV_x + U_x^3 \\ &-4U_x^2V + 6U_xV^2 - \tfrac{4}{3}U^3U_x - 4V^3 + \tfrac{8}{3}U^3V\big] + \big(\tfrac{1}{3}\alpha\big)^2\big[3D(2V - U_x) \\ &-C(V_x + U^2) + a\big(2VV_x - U_xV_x - U^2U_x + 2U^2V\big)\big] + a^2C. \end{aligned} \tag{57}$$

The expression for B, which is too long to reproduce here, is of degree three in U_{xx} and two in V_{xx}.

Under the reduction $(U, V) = (2W, W_x)$, with $C = 0$ (since we integrated equations (48) and (49), with $g_0 = 0$), and with $g_1 = q_1/9$, the system (52) and (53) reduces to the single ODE

$$\alpha\big(W_{xxxx} + 12WW_{xx} + 6W_x^2 + \tfrac{32}{3}W^3\big) + 3a\big(W_{xx} + 4W^2\big) - q_1x + 9D = 0. \tag{58}$$

This ODE is equivalent to an ODE found by Cosgrove using Painlevé classification [11], and which is thought to define a new transcendent. The above matrix linear problem (54) and (55) then reduces to the matrix linear problem for this ODE given in [10] (see also [24]). In the case $q_1 = 0$ we find that for this reduction the above expression for A vanishes, and the expression for B then gives a first integral for (58) (with $q_1 = 0$) of degree two in W_{xxx}. This is presumably equivalent (modulo an additional simple first integral) to that given in [11].

We note that it is possible to eliminate V from the system (52) and (53) to obtain a sixth-order ODE in U of degree two, which also has of course solutions given in terms of solutions of (58). We also note that this system (52) and (53) arises here as the (integrated) stationary flow of an integrable evolution equation (similarly its generalization (48) and (49)), although the reduction (58) does not appear to do so.

6. Conclusions

Non-isospectral extensions to 2 + 1 dimensions of the third-order scattering problem $\psi_{xxx} + U\psi_x + (V - \lambda)\psi = 0$ have not been considered in the literature before. Here we have given such an extension. This then allows the construction of a new integrable variant of the Boussinesq hierarchy in 2 + 1 dimensions, together with its corresponding hierarchy of underlying linear problems. This extends our previous work on non-isospectral extensions of the Sawada–Kotera and Kaup–Kupershmidt hierarchies. Reductions of this non-isospectral Boussinesq hierarchy to lower dimensions then include non-autonomous extensions of Boussinesq and inverse Boussinesq flows, and also new hierarchies of systems of ODEs, all together with their underlying linear problems. In the general case these hierarchies of PDEs and ODEs are non-local. In the local case we have identified one hierarchy of ODEs as a P_{IV} hierarchy, and another as being based on a generalization of Cosgrove's equation. It is this generalization of Cosgrove's equation, i.e. (52) and (53), which has in turn the further generalization (48) and (49), which is the simplest of our new examples.

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